Properties of Toeplitz operators on analytic function spaces: from function symbols to distributions

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Academic dissertation

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Helsinki, August 2011

Antti Perälä

List of contributed articles

This doctoral dissertation consists of an introductory part, the following five refereed journal articles and a notes section including some complementary results and errata. The author acknowledges the respective journals as the original publishers of these articles.

[A] A. Perälä, J. Taskinen, J. Virtanen, Toeplitz operators with distributional symbols on Bergman spaces. Proceedings of the Edinburgh Mathematical Society, Volume 54, Issue 02, 505–514, (2011).

[B] A. Perälä, J. Taskinen, J. Virtanen, New results and open problems on Toeplitz operators in Bergman spaces. New York Journal of Mathematics, Volume 17a, 147–164, (2011).

[C] A. Perälä, J. Taskinen, J. Virtanen, Toeplitz operators with distributional symbols on Fock spaces. Functiones et Approximatio, Commentarii Mathematici, Volume 44, Number 2, 203–213, (2011).

[D] A. Perälä, Toeplitz operators on Bloch-type spaces and classes of weighted Sobolev distributions. Integral Equations and Operator Theory, Volume 71, Number 1, 113–128 (2011).

[E] A. Perälä, J. Virtanen, A note on the Fredholm properties of Toeplitz operators on weighted Bergman spaces with matrix-valued symbols. Operators and Matrices, Volume 5, Issue 1, 97–106, (2011).

1. Introduction

Toeplitz operators are among the most important types of concrete operators with applications to several branches of mathematics including mathematical physics, complex analysis, theory of normed algebras, dynamical systems, random matrix theory and operator theory. They have been even used in areas such as control theory and fluid dynamics. Despite their apparent simplicity, Toeplitz operators often provide non-trivial examples and counter-examples to many phenomena of mathematics. A good account on applications is provided in the books [7], [8] and [28].

In this thesis we study boundedness and compactness criteria for Toeplitz operators by extending the definition from function symbols to distributional ones. This kind of approach takes into account the behaviour of the derivatives of analytic functions in a convenient way. We will also look at the Fredholm properties of Toeplitz operators with matrix-valued symbols and study their index theory.

We will start by reviewing what is generally known about the boundedness, compactness and Fredholm theory of Toeplitz operators having function symbols. We will discuss Bergman, Fock and Bloch-type spaces, but the main focus is on the Bergman space case. The reasons for this are the following: the Bergman space setting is the archetypal setting for our purposes, while the Bloch and Fock space results are often analogous when appropriate (Gaussian, polynomial or logarithmic) weights are applied. On the other hand, most of our results are for Bergman spaces, anyway. An interested reader can find plenty of material on Bloch and Fock spaces in the references. Our approach to the distributional symbol case is explained in the second part of this thesis.

We have taken the liberty to combine several results from one source to one theorem. This does not always reflect the course of thinking leading to the results, but makes the exposition more readable. Detailed references to the corresponding partial results are provided for the convenience of the reader.

The notation is standard: by $\mathbb{D}$ be mean the open unit disk of the complex plane $\mathbb{C}$, centered at the origin, and $\mathbb{T}$ is used for the boundary of $\mathbb{D}$, $\mathbb{T} = \partial\mathbb{D}$. More notation is explained along the way.

1.1. Preliminaries

We will work either on Bergman, Bloch or Fock spaces. For general reference on these spaces, we mention [14, 36, 37] for Bergman spaces, [14, 35, 36, 37] for Bloch spaces and [12, 15, 18] for Fock spaces. Let $0 < p < \infty$, then the Bergman space A^p of the unit disk consists of those analytic $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\|f\|_p := \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty,$$

the space H^∞ consists of bounded analytic functions on $\mathbb{D}$. When $p \geq 1$, we can equip these spaces with respective L^p-norms making them Banach spaces.

The Bergman projection P is an integral operator induced by the Bergman kernel K_z;

$$Pf(z) = \int_{\mathbb{D}} f(w)K_z(w)dA(w) = \int_{\mathbb{D}} f(w)(1-z\bar{w})^{-2}dA(w) \ , \ z \in \mathbb{D}. \quad (1.1)$$

The projection P is bounded $L^p \to A^p$ whenever $p \in (1,\infty)$. It is not bounded $L^1 \to A^1$ or $L^\infty \to H^\infty$, but the definition still makes sense. If $a \in L^\infty$, we can define the Toeplitz operator with symbol a by

$$T_a f(z) = P(af)(z) = \int_{\mathbb{D}} \frac{f(w)a(w)dA(w)}{(1-z\bar{w})^2} \ , \ z \in \mathbb{D}. \quad (1.2)$$

Here f can be assumed to be a member of A^p for $1 \le p < \infty$ or H^∞.

If $1 < p < \infty$, then the dual space of A^p can be identified with the Bergman space A^q where q is the Hölder conjugate of p; that is, $1/p+1/q = 1$. The dual space of A^1 can be identified with the Bloch space $\mathcal{B}$ consisting of those analytic $f : \mathbb{D} \to \mathbb{C}$ for which the Bloch seminorm:

$$\|f\|_* := \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)|$$

is finite. Equipped with the norm

$$\|f\|_{\mathcal{B}} = \|f\|_* + |f(0)| \ , \ f \in \mathcal{B},$$

the Bloch space becomes a Banach space. The Bergman projection (1.1) is bounded from L^∞ onto $\mathcal{B}$. The formula (1.2) makes sense also when $f \in \mathcal{B}$. We will also consider the general Bloch-type spaces $\mathcal{B}_d$ for $d > 0$, they are defined similarly to $\mathcal{B}$ by using the semi-norm

$$\|f\|_{*,d} := \sup_{z\in\mathbb{D}} (1-|z|^2)^d |f'(z)|.$$

It is possible to define Toeplitz operator on $\mathcal{B}_d$, but then one has to be careful; for large values of d the condition $a \in L^\infty$ is not sufficient for T_a to be defined.

Let $1 \le p < \infty$ and $\gamma > 0$. Then the Fock space F^p_γ of the complex plane consists of all entire functions f for which

$$\|f\|_{p,\gamma} := \left(\int_{\mathbb{C}} |f(z)|^p e^{\frac{-\gamma p|z|^2}{2}} dA_{p,\gamma}(z) \right)^{1/p}$$

is finite. Here $dA_{p,\gamma}(z) = c_{p,\gamma} dA(z)$ is the Lebesgue area measure normalized so that

$$\int_{\mathbb{C}} e^{-(\gamma p/2)|z|^2} dA_{p,\gamma}(z) = 1.$$

The space F^∞_γ consists of all entire functions for which the norm

$$\|f\|_{\infty,\gamma} := \sup_{z\in\mathbb{D}} |f(z)| e^{\frac{-\gamma|z|^2}{2}}$$

is finite. The Fock spaces closed subspaces of the respective L^p_γ spaces defined in the usual way. The Fock projection P is then the integral operator induced by the Fock kernel K_z as follows

$$\begin{aligned} Pf(z) &= \int_{\mathbb{C}} f(w)K_z(w)e^{-\gamma|w|^2}dA_{p,\gamma}(w) \\ &= \int_{\mathbb{C}} f(w)e^{\gamma z\bar{w}}e^{-\gamma|w|^2}dA_{p,\gamma}(w)\ ,\ z\in\mathbb{C}. \end{aligned}$$

We note that since we deal with the Bergman and Fock spaces separately, there is no confusion in the notation. The Fock projection is known to be bounded $L^p_\gamma \to F^p_\gamma$ for each $p\in[1,\infty]$. If $a\in L^\infty$, we can then define the Toeplitz operator analogously to (1.2) by the formula

$$T_af(z) = P(af)(z) = \int_{\mathbb{C}} a(w)f(w)e^{\gamma z\bar{w}}e^{-\gamma|w|^2}dA_{p,\gamma}(w)\ ,\ z\in\mathbb{C}. \qquad (1.3)$$

1.2. Boundedness and compactness of Toeplitz operators

Boundedness and compactness are among the most fundamental properties of linear operators. We recall that an operator T between Banach spaces X and Y is bounded if $T(E)\subset Y$ is bounded for each bounded $E\subset X$. The norm of T is then $\|T\| = \sup_{x\in X,\|x\|=1}\|Tx\|$. The operator T is compact if the closure of $T(E)$ is compact in Y.

Hardy spaces. Suppose that $1<p<\infty$, $H^p(\mathbb{T})\subset L^p(\mathbb{T})$ is the Hardy space and $P: L^p(\mathbb{T})\to H^p(\mathbb{T})$ is the Riesz projection. If a is a measurable function, M_a is the related multiplication operator and T_a is the Toeplitz operator:

$$T_af = PM_af = P(af),$$

then T_a is bounded if and only if $a\in L^\infty(\mathbb{T})$. Moreover, T_a is compact if and only is a is the zero function. Theory of Toeplitz operators on Hardy spaces is documented in several sources, we mention [8] and [13].

Although the boundedness and compactness questions have been settled in the Hardy space case, they are still open in many other spaces, including the Bergman, Bloch and Fock spaces. We mention that finite rank Toeplitz operators on the Bergman space setting were recently characterized by D. Luecking in [20].

Carleson measures. Carleson measures have been invaluable tool in several aspects of mathematical analysis. They are named after L. Carleson who developed them in Hardy space setting in the course of his famous proof of the Corona theorem, see [9]. The boundedness and compactness of Toeplitz operators can be approached via the use of similar measures in other function spaces.

A non-negative Borel measure μ on $\mathbb{D}$ is said to be Carleson measure for Bergman spaces if there exists $C>0$ such that

$$\int_{\mathbb{D}}|f(z)|^pd\mu(z) \le C\int_{\mathbb{D}}|f(z)|^pdA(z)$$

holds for every $f \in A^p$. Equivalently, the embedding $i_p : A^p \to L^p(\mu)$ is required to be bounded. Here $p \in [1, \infty)$ can be chosen freely and we always get the same collection of measures. Denote by β the Bergman metric of $\mathbb{D}$ and write $B_\beta(z, r) := \{w \in \mathbb{D} : \beta(z, w) < r\}$ for the Bergman disk of radius $r > 0$ centered at $z \in \mathbb{D}$. Geometric characterizations for Carleson measures have been studied by several authors, including W. Hastings, D. Luecking, V. L. Oleinik, B. S. Pavlov and K. Zhu, see [16, 19, 21, 33]. In what follows, we refer to the work of Zhu.

Proposition 1.2.1. (Theorem 7 of [33]) *Let $r > 0$. A non-negative Borel measure μ on $\mathbb{D}$ is a Carleson measure on Bergman spaces if and only if there exists $C > 0$ such that*

$$\mu(B(z, r)) \leq C|B_\beta(z, r)|$$

for every $z \in \mathbb{D}$.

Here, and in what follows, we use $|A|$ to denote the normalized Lebesgue measure of A. Note that by the virtue of above proposition, the radius $r > 0$ can be chosen freely and we always end up with the same collection of measures.

We can also define vanishing Carleson measures on the Bergman spaces by requiring that the embedding i_p is compact for some (and thus all) $p \in (1, \infty)$. Note that $p = 1$ is not included in the definition. A characterization using $p = 1$ is also possible, see [37]. Again, being vanishing Carleson measure on Bergman spaces is characterized by the following:

Proposition 1.2.2. (Theorem 11 of [33]) *Let $r > 0$. A non-negative Borel measure μ on $\mathbb{D}$ is a vanishing Carleson measure on Bergman spaces if and only if*

$$\lim_{|z| \to 1^-} \mu(B_\beta(z, r))/|B_\beta(z, r)| = 0.$$

Again, every choice of $r > 0$ in the above proposition will result in the same collection of measures.

In the above proposition the sets $B_\beta(z, r)$ can be replaced by squares to be defined in (1.4); these squares are actually often referred as Carleson squares. It is often up to the person using Carleson measures whether to use Bergman disks or Carleson squares; the former provides conformal invariance, the latter is easier when integrating in polar coordinates.

Carleson measures can be defined in Fock spaces, as well. The definition is analogous, the Bergman disks are replaced by Euclidean disks and the Carleson squares by ordinary ones.

It is known that a non-negative measure will generate bounded Toeplitz operator on A^p $(1 < p < \infty)$ if and only if it is a Carleson measure on Bergman spaces. A result of similar nature is true about Fock spaces. Also, compactness is equivalent to being vanishing Carleson on the respective space. We refer the reader to the sources [18, 19, 23, 33, 37] for detailed analysis regarding Carleson measures. If a is a non-negative integrable function, it is natural to ask whether the measure $a(z)dA(z)$ is Carleson. The same goes for the Fock spaces and the non-negative Borel measures on the complex plane.

For locally integrable functions of the unit disk, a more general result was recently obtained in [26]. Denote by $\mathcal{D}$ the family of sets $D := D(r,\theta)$ defined as

$$D = \{\rho e^{i\phi} : r \leq \rho < 1 - (1-r)/2, \theta \leq \phi \leq \theta + \pi(1-r)\}. \tag{1.4}$$

Given $a \in L^1_{loc}$, $D \in \mathcal{D}$ and $\zeta = \rho e^{i\phi} \in D$ we denote

$$\hat{a}_D(\zeta) := \frac{1}{|D|}\int_r^\rho \int_\theta^\phi a(se^{i\psi})sd\psi ds. \tag{1.5}$$

For $D \in \mathcal{D}$ we also denote by $d(D)$ the distance from the set D to the boundary of the unit disk. The main theorems of [26] are as follows:

Theorem 1.2.3. (Theorem 2.3 and Theorem 2.6 of [26]) *Let $1 < p < \infty$ and $a \in L^1_{loc}$. Suppose that there exists $C > 0$ such that*

$$|\hat{a}_D(\zeta)| \leq C \tag{1.6}$$

for each $D \in \mathcal{D}$ and $\zeta \in D$. Then $T_a : A^p \to A^p$ is well-defined and bounded. Moreover, there exists a $C' > 0$ such that

$$\|T_a : A^p \to A^p\| \leq C' \sup_{D\in\mathcal{D},\zeta\in D} |\hat{a}_D(\zeta)|.$$

If, in addition, we have

$$\lim_{d(D)\to 0} \sup_{\zeta\in D} |\hat{a}_D(\zeta)| = 0, \tag{1.7}$$

then T_a is compact on A^p.

Note that the above theorem does not assume that a is positive and it also allows locally integrable symbols.

Berezin transform. One very important tool in the study of operators on Bergman and Fock spaces is the Berezin transform. The idea is from the paper [4] of F. A. Berezin. The Berezin transform is particularly powerful tool in the study of Toeplitz operators and elements in Toeplitz algebras. Let $a \in L^1(\mathbb{D})$, then the Berezin transform $\tilde{a}$ of a can be defined as

$$\tilde{a}(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^2 a(w)dA(w)}{|1-z\bar{w}|^4}. \tag{1.8}$$

The integral kernel of the above operator is in fact $|k_z(w)|^2$ where $k_z(w) = K_z(w)/\|K_z\|_2$, as one can easily verify. For general $p \in (1,\infty)$, and an operator T whose domain contains all of H^∞ the definition for Berezin transform becomes

$$\tilde{T}(z) = \langle Tk_z^{(q)}, k_z^{(p)}\rangle,$$

where $1/p + 1/q = 1$,

$$k_z^{(p)}(w) = \frac{(1-|z|^2)^{2/q}}{|1-z\bar{w}|^2}$$

and $\langle \cdot, \cdot \rangle$ stands for the integral pairing of spaces A^p and A^q. We set $\tilde{a}(z) = \tilde{T}_a(z)$ and the new definition then agrees with (1.8) above when $a \in L^1$. The reader might want to consult [24] for more details on the general case, in particular when not dealing with Hilbert space operators.

For a positive L^1 symbol a, it is known that the Berezin transform of a is bounded if and only if $a(z)dA(z)$ is a Carleson measure if and only if T_a is bounded on A^p for $1 < p < \infty$. Similarly, compactness can be formulated in terms of vanishing of the Berezin transform near the boundary. These result are proven in [37] for $p = 2$, for other values of p, these result follow from those in [26]. Also, N. Zorboska has improved this result for the space A^2, see [38].

The Berezin transform is exceptionally useful when studying compactness of Toeplitz operators. In 1998 S. Axler and D. Zheng proved that for $a \in L^\infty$ the Berezin transform of a satisfies

$$\lim_{|z|\to 1^-} \tilde{a}(z) = 0$$

if and only if the Toeplitz operator T_a is compact on A^2. This result was proven for more general symbols by N. Zorboska in [38]. For our purposes the most convenient result is due to D. Suárez, see [24].

Theorem 1.2.4. (Theorem 9.5 of [24]) *Let $1 < p < \infty$. Suppose that the operator T is contained in the Banach algebra of operators on A^p generated by all T_a with $a \in L^\infty$. Then T is compact on A^p if and only if*

$$\tilde{T}(z) \to 0,$$

as $|z| \to 1^-$.

In the Fock space setting, the Berezin transform can be defined similarly, by inner product, or by using the integral formula like in (1.8). There also exists a more general concept, known as the heat-semigroup, which is a powerful tool for studying Toeplitz operators, see [3, 6].

1.2.1. The Bloch-type spaces and A^1. By duality, the boundedness and compactness results for Toeplitz operators on A^1 are closely related to those of $\mathcal{B}$. The Bloch space can be considered as an analytic version of BMO_∂, the space of bounded mean oscillation in Bergman metric. The Bergman projection is bounded $BMO_\partial \to \mathcal{B}$, and therefore the question of boundedness can be approached by finding the pointwise multipliers of BMO_∂. The reader should consult [34, 35] for reference. A general result for boundedness was recently obtained by Taskinen and Virtanen in [27].

Write $\mathcal{W}(z) = 1 - \log(1 - |z|)$ for the logarithmic weight (a slightly different logarithmic weight is used in papers [30] and [D], but actually both weights will do here). Denote by X the set of locally integrable symbols a satisfying

$$|\hat{a}_D(\zeta)| \leq C/\mathcal{W}(\zeta)$$

for some $C > 0$ and all $D \in \mathcal{D}$ and $\zeta \in D$ (see (1.4) and (1.5)).

By $BO_{\log}$ we mean the logarithmically weighted space of bounded oscillation. A continuous function $f : \mathbb{D} \to \mathbb{C}$ belongs to the space $BO_{\log}$ if there exists a positive constant C such that

$$\sup_{w \in B_\beta(z,1/2)} \mathcal{W}(z)|f(z) - f(w)| \leq C$$

for every $z \in \mathbb{D}$.

Theorem 1.2.5. (Theorem 7 of [27]) *Suppose $a \in BO_{\log} + X$. Then T_a is well-defined and bounded on A^1.*

Proposition 1.2.6. (Proposition 9 of [27]) *Suppose $a \in X$ and that*

$$\lim_{d(D)\to 0} \sup_{D\in\mathcal{D},\zeta\in D} |\hat{a}_D(\zeta)|\mathcal{W}(\zeta) = 0,$$

then T_a is compact on A^1.

Both of the results also hold for the classical Bloch space $\mathcal{B}$, by duality.

A different approach to the boundedness and compactness of Toeplitz operators on the Bloch spaces $\mathcal{B}_d$ for $d \in (0, \infty)$ and A^1 was given in 2006 by Z. Wu, R. Zhao and N. Zorboska. We only deal with $\mathcal{B}$ and A^1, for details about $\mathcal{B}_d$ with $d \neq 1$, we refer the reader to [30]. The results in their full generality involve measures. We, however, only present the results under the assumption that $a \in L^\infty$ (for a general measure, the condition involves a rather technical size estimate).

By the logarithmic Bloch space $\mathcal{LB}$ we mean the space of functions f analytic on $\mathbb{D}$ satisfying

$$\sup_{z\in\mathbb{D}}(1 - |z|^2) \log(2/(1 - |z|^2))|f'(z)| < \infty.$$

Corollary 1.2.7. (Corollary 2.1 and Corollary 2.2 of [30]) *Suppose $a \in L^\infty$. Then $T_a : \mathcal{B} \to \mathcal{B}$ is bounded if and only if $P(a) \in \mathcal{LB}$. It follows that $T_a : A^1 \to A^1$ is bounded if $P(\bar{a}) \in \mathcal{LB}$.*

Analogous compactness results are also proven, see Theorem 3.1 of [30].

1.3. Fredholm theory

Fredholm operators appear often in applications and their properties are of crucial importance when studying integral equations or even abstract index theory. Toeplitz operators possess a very rich Fredholm theory with beautiful results on the index and essential spectra.

Recall that an operator T on a Banach space X is Fredholm if

$$n(T) := \dim\ker T < \infty \text{ and } d(T) := \dim X/TX < \infty.$$

The index of T is defined by the formula

$$\operatorname{Ind} T = d(T) - n(T).$$

Equivalently, T is Fredholm if and only if it is invertible in the Calkin algebra of all bounded operators modulo compact operators; that is, there exists a bounded operator S such that

$$TS = I + K_1 \text{ and } ST = I + K_2$$

for some compact operators K_1 and K_2.

Hardy spaces. The Fredholm theory for Toeplitz operators in Hardy spaces is very well understood. An extensive treatment of this field is found in [8]. Fredholm theory in Hardy spaces is also significantly different from Bergman spaces. Reasons for this are the lack of effective factorization in Bergman spaces and the general difficulty when dealing with the unit disk, instead of the unit circle, for instance. All of the results of this section are well-known in the setting of Hardy spaces, when understood correctly.

Reflexive Bergman spaces. The most natural Bergman space on which Fredholm theory can be established is the Hilbert space A^2. Results in this direction are, among others, due to U. Venugopalkrishna and L. Coburn in [29] and [11], respectively. When letting $1 < p < \infty$ we deal with reflexive Bergman spaces, where we still have a bounded Bergman projection, which makes the course of arguments much easier compared to the A^1 theory.

Let us first consider the space A^2. The algebra $T(C)$ is the Banach algebra generated by Toeplitz operators with symbols in $C(\overline{\mathbb{D}})$ (it is also a C^*-algebra of operators on A^2).

A beautiful result of L. Coburn in [11] goes as follows:

Theorem 1.3.1. (Theorem 1 and Corollary to Theorem 1 of [11]) *The algebra $T(C)$ is irreducible and contains all compact operators. Each element $T \in T(C)$ satisfies*

$$T = T_a + K \tag{1.9}$$

for some $a \in C(\overline{\mathbb{D}})$ and compact operator K. Moreover, T_a is Fredholm if and only if $a(z) \neq 0$ for all $z \in \mathbb{T}$.

This result was generalized to reflexive Bergman space in 1992 by X. Zeng, see [31]. Zeng also showed that the commutator ideal of $T(C)$ is equal to the ideal of compact operators.

The equation (1.9) is a consequence of the fact that for $a, b \in C(\overline{\mathbb{D}})$, the semi-commutator $T_{ab} - T_a T_b$ is compact. The largest C^*-subalgebra of L^∞ for which this is true was discovered by K. Zhu in his paper [32]. Let us call this algebra Q. The main result by Zhu is the following.

Theorem 1.3.2. (Proposition 6 and Theorem 13 of [32]) *Let $a \in L^\infty$. The following are equivalent:*

(i) $a \in Q$;

(ii) $a \in VMO_\partial \cap L^\infty$;

(iii) The Hankel operators H_a and $H_{\bar{a}}$ are both compact $A^2 \to L^2$.

In particular, we have that $Q = VMO_\partial \cap L^\infty$.

In his paper Zhu also proves that if $a \in Q$, then the Berezin transform of $a - \tilde{a}$ vanishes when approaching the boundary. In what follows, if a is a function defined on $\mathbb{D}$ and $0 < r < 1$, we will use the notation

$$a_r := a_r(\theta) = a(re^{i\theta}).$$

If a is defined on $\overline{\mathbb{D}}$, we will also write $a_1(\theta) := a(e^{i\theta})$. By $\operatorname{Ind} a_r$ $(r \in (0,1])$we mean the winding number of the curve $a_r(\theta)$ around zero. Since the Berezin transform of $a \in Q$ behaves nicely, the following consequence now follows.

Theorem 1.3.3. (Theorem 15 of [32]) *Let $a \in Q$. Then $T_a : A^2 \to A^2$ is Fredholm if and only if $T_{\tilde{a}}$ is Fredholm if and only if there exists $\epsilon > 0$ and $r < 1$ such that $|\tilde{a}(z)| > \epsilon$, whenever $r < |z| < 1$. The Fredholm index of T_a is then given by the formula*

$$\operatorname{Ind} T_a = -\operatorname{Ind} \tilde{a}_R,$$

where $r < R < 1$.

A description of the essential spectrum of T_a was also given; denote by $\beta\mathbb{D}$ the Stone-Čech compactification of $\mathbb{D}$ (see [17]) and write $\tilde{a}^*$ for the unique extension of $\tilde{a}$ to $\beta\mathbb{D}$. We then have

$$\sigma_{ess} T_a = \tilde{a}^*(\beta\mathbb{D} \setminus \mathbb{D}).$$

In this thesis we will also consider the symbol class $H := C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})$, the so-called Douglas algebra. The class H is a natural class in the Hardy space setting, and gives a nice generalization for $C(\overline{\mathbb{D}})$ in the Bergman space setting, as well. An obvious difficulty, like in the case of Q, is then the lack of continuous boundary values.

The following theorem was proven in 1997 by B. R. Choe and Y. J. Lee.

Theorem 1.3.4. (Theorem 1 and Lemma 6 of [10]) *Let $1 < p < \infty$ and $a \in H$. The operator T_a is Fredholm if and only if there exist $\epsilon > 0$ and $r < 1$ such that $|a(z)| > \epsilon$, whenever $r < |z| < 1$. The essential spectrum is given by*

$$\sigma_{ess} T_a = a^*(\beta\mathbb{D} \setminus \mathbb{D}),$$

where a^* is the unique extension of a to $\beta\mathbb{D}$.

In their paper Choe and Lee also study Toeplitz operators on the unit ball of $\mathbb{C}^n$. The Fredholm criterion is analogous. They also prove that the Fredholm index is always zero when $n \geq 2$. An index formula for the case $n = 1$ is given in [E].

A natural class in the Bergman space setting is also the algebra of piece-wise continuous function. This involves joining the discontinuities with suitable curves. We will not deal with piece-wise continuous symbols in this thesis. A rather complete treatment of essential spectra and index theory in the setting of A^2 is given in [28]. To our knowledge, most of the questions about piece-wise continuous symbols are still open for A^p when $p \neq 2$.

Bergman space A^1. When dealing with Toeplitz operators on the Bergman space A^1, one immediately runs into one serious problem. The Bergman projection is not bounded $L^1 \to A^1$. It follows that $a \in L^\infty$ is not sufficient for the boundedness of T_a. Conditions for boundedness are given in [25], [27] and [30], for instance.

The Fredholm theory for Toeplitz operators on A^1 was studied in 2008 by J. Taskinen and J. Virtanen. They establish Fredholmness criterion and index-theorem for Toeplitz operators having symbols in

$$CL := C(\overline{\mathbb{D}}) \cap VMO_{\partial \log},$$

see [25] for more details. The following theorem, proven in [25], sums up the reason, why CL works fine in this setting.

Theorem 1.3.5. (Theorem 6, Corollary 8 and Theorem 10 of [25]) *Suppose $a \in CL$. Then T_a is bounded $A^1 \to A^1$ and H_a is compact $A^1 \to L^1$. If, in addition, $a(z) = 0$ for each $z \in \mathbb{T}$, then T_a is compact on A^1.*

The above theorem makes it possible to treat $T_a : A^1 \to A^1$ in a fashion similar to the reflexive Bergman space case. The following theorem was proven.

Theorem 1.3.6. (Theorem 11 and Theorem 12 of [25]) *Suppose $a \in CL$. Then $T_a : A^1 \to A^1$ is Fredholm if and only if $a(z) \neq 0$ for all $z \in \mathbb{T}$. Moreover, the index of T_a is then given by the formula*

$$\operatorname{Ind} T_a = -\operatorname{Ind} a_1.$$

2. Properties of Toeplitz operators on analytic function spaces: from function symbols to distributions

This thesis consists of five articles. The following sections serve as short introductions to these articles. The order of the section is not the same as the chronological order of the respective articles. We made the choice to put the theory of distributional symbols in the beginning, even though the paper [E] was actually the first one to be finished. The article [B], however, contains a short section on Fredholm theory on the Bergman space A^1. When presenting results in the summaries of the articles, we always refer to results of the particular article in question.

The notation in the article [E] agrees with the usual notation, which is used in the introductory part. In the other four articles, to avoid confusion, we will use the following conventions. Suppose that X and Y are normed spaces. For an element $x \in X$ we denote by $\|x; X\|$ the norm of x in X. Similarly, $\|T : X \to Y\|$ is used for the operator norm of a bounded operator $T : X \to Y$.

2.1. Article A: Toeplitz operators with distributional symbols on Bergman spaces

In this article we deal with the problem of extending the definition of Toeplitz operators to distributional symbols. Our approach involves symbols that are not necessarily compactly supported distributions. Some results about Toeplitz operators with compactly supported distributional symbols can be found in [1]. We show how the developed machinery also goes to produce unbounded function symbols that generate bounded (or even compact) Toeplitz operators.

The symbol classes $W_\nu^{-m,\infty}$. Natural distributional symbol classes for Toeplitz operators on Bergman spaces are the weighted distributional Sobolev spaces $W_\nu^{-m,\infty}$. Let $m \in \mathbb{N}$. Then the weighted Sobolev space $W_\nu^{m,1}$ consists of functions f measurable of $\mathbb{D}$, for which the norm

$$\|f; W_\nu^{m,1}\| := \sum_{|\alpha| \le m} \int_{\mathbb{D}} |D^\alpha f(w)| \nu(w)^{|\alpha|} dA(w) \tag{2.1}$$

is finite. Here $\nu(z) = 1 - |z|^2$ is the standard weight and we use the multi-index-notation for α, $|\alpha|$ and D^α. It is easy to see that $A^1 \subset W_\nu^{m,1}$ for each $m \in \mathbb{N}$.

Definition 2.1.1. Let $m \in \mathbb{N}$. The weighted distributional Sobolev space $W_\nu^{-m,\infty}$ consists of those distributions $a \in \mathcal{D}'$ that admit a representation

$$a = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.2}$$

where each b_α belongs to L_α^∞; that is, $\|b_\alpha; L_\alpha^\infty\| := \|\nu^{-|\alpha|} b_\alpha\|_\infty < \infty$.

The derivative D^α in the above theorem is understood in the distributional sense. We will equip $W_\nu^{-m,\infty}$ with the norm

$$\|a; W_\nu^{-m,\infty}\| := \inf \max_{|\alpha|\leq m} \|b_\alpha; L_\alpha^\infty\|,$$

where the infimum is taken over all possible representations (2.2). We prove the following:

Lemma 2.1.2. (Lemma 2.2 and Lemma 2.4) *Let $m \in \mathbb{N}$. Then the space of compactly supported test-functions C_0^∞ is dense in the Sobolev space $W_\nu^{m,1}$. Moreover, the dual space of $W_\nu^{m,1}$ is isometrically isomorphic to $W_\nu^{-m,\infty}$ under the dual pairing*

$$\langle f, a\rangle := \sum_{|\alpha|\leq m} \int_{\mathbb{D}} D^\alpha f(w) b_\alpha(w) dA(w) \ , \ f \in W_\nu^{m,1} \ , \ a \in W_\nu^{-m,\infty}.$$

We note that the representation (2.2) is usually not unique, but each representation will give the same value for the dual pair $\langle\cdot,\cdot\rangle$ above.

The distributional classes $W_\nu^{-m,\infty}$ contain all the compactly supported distributions in the following sense. If $a \in \mathcal{D}'$ is a compactly supported distribution, then there exists $m \in \mathbb{N}$ such that $a \in W_\nu^{-m,\infty}$. This follows from the standard distribution theory, see for instance [22].

Boundedness and compactness of Toeplitz operators with distributional symbols. The definition for Toeplitz operator with symbol $a \in W_\nu^{-m,\infty}$ goes as follows.

Definition 2.1.3. Suppose $a \in W_\nu^{-m,\infty}$ for some $m \in \mathbb{N}$. The Toeplitz operator T_a is then defined as

$$T_a f(z) := \sum_{|\alpha|\leq m} \int_{\mathbb{D}} D^\alpha \left(\frac{f(w)}{(1-z\bar{w})^2}\right) b_\alpha(w) dA(w) \ , \ z \in \mathbb{D}. \qquad (2.3)$$

Here (b_α) is a collection of functions satisfying (2.2).

The value of the above operator does not depend on the representation for a. It also agrees with the standard definition for Toeplitz operator with L^∞ symbol. The following two are the main theorems of this paper.

Theorem 2.1.4. (Theorem 3.1) *Suppose $a \in W_\nu^{-m,\infty}$ for some $m \in \mathbb{N}$. Then the Toeplitz operator T_a is bounded on A^p for $1 < p < \infty$. Moreover, there exists $C := C(m,p) > 0$ such that*

$$\|T_a : A^p \to A^p\| \leq C\|a; W_\nu^{-m,\infty}\|.$$

Theorem 2.1.5. (Theorem 4.2) *Suppose $a \in W_\nu^{-m,\infty}$ for some $m \in \mathbb{N}$ and a has a representation*

$$a = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha b_\alpha,$$

where each b_α satisfies

$$\operatorname{ess} \lim_{r\to 1^-} \sup_{r<|z|<1} \nu^{-|\alpha|}(z)|b_\alpha(z)| = 0.$$

Then T_a is compact on A^p for $1 < p < \infty$.

2.2. Article B: New results and open problems on Toeplitz operators in Bergman spaces

This article complements papers [A], [E], [25] and [26]. We also present open questions related to Toeplitz operators on Bergman spaces.

Locally integrable symbols. We prove that if $a \in L^1_{loc}$ is radial, then the condition in [26] can be simplified. Suppose that $a \in L^1_{loc}$ is radial: $a(z) = a(|z|)$. For all $r \in (0,1)$ let $I := I(r) = [r, 1-(1-r)/2]$ and

$$\hat{a}_I(\rho) = \frac{1}{1-r}\int_r^\rho a(s)s ds, \tag{2.4}$$

where $\rho \in I$.

The following lemma connects (2.4) to the main result of [26].

Lemma 2.2.1. (Lemma 2 and Corollary 3) *Suppose $a \in L^1_{loc}$ is radial. Then a satisfies the condition* (1.6) *if and only if there exists $C > 0$ such that*

$$\sup_{r\in(0,1)} \sup_{\rho\in I(r)} |\hat{a}(\rho)| \le C.$$

Moreover, if $1 < p < \infty$, then there exists $C_p > 0$ such that

$$\|T_a : A^p \to A^p\| \le C_p \sup_{r\in(0,1)} \sup_{\rho\in I(r)} |\hat{a}(r)|.$$

Similarly, we can establish a compactness result.

Lemma 2.2.2. (Lemma 8) *For a radial $a \in L^1_{loc}$ the condition*

$$\lim_{r\to 1^-} \sup_{\rho\in I(r)} |\hat{a}(\rho)| = 0 \tag{2.5}$$

is equivalent with the vanishing condition (1.7)*.*

It also follows that if $a \in L^1_{loc}$ satisfies (2.5), then the associated Toeplitz operator T_a is compact on A^p for $1 < p < \infty$.

Distributional symbols. We want to find a radial counterpart of the $W_\nu^{-m,\infty}$ classes defined in [A]. Let $\mu(r) = r(1-r^2)$ $(r \in (0,1))$ be a radial weight function. Let $m \in \mathbb{N}$. We can then define $W_\mu^{-m,\infty}(0,1)$ as the set of those distributions a on $(0,1)$ that have a representation

$$a = \sum_{0\le j\le m} (-1)^j D^j b_j, \tag{2.6}$$

where each b_j satisfies $\|b_j; L_j^\infty\| := \|\mu^{-j} b_j\|_\infty < \infty$. We can make $W_\mu^{-m,\infty}(0,1)$ a Banach space by equipping it with the norm

$$\|a; W_\mu^{-m,\infty}(0,1)\| = \inf \max_{0\le j\le m} \|b_j; L_j^\infty\|,$$

where the infimum is taken over all possible representations (2.6).

It can be seen that for each $m \in \mathbb{N}$ we have $W_\mu^{-m,\infty}(0,1) \subset W_\nu^{-m,\infty}$ if we understand $b_j(z) = b_j(|z|)$. Hence every $a \in W_\mu^{-m,\infty}(0,1)$ can be seen to generate a bounded Toeplitz operator on A^p for $1 < p < \infty$. We also prove:

Proposition 2.2.3. (Proposition 7) *Suppose $a \in L^1_{loc}$ is radial, a vanishes on a neighbourhood of* 0, *and that there exists $C > 0$ such that*

$$\sup_{r\in(0,1)} \sup_{\rho\in I(r)} |\hat{a}(\rho)| \leq C.$$

Then there exists $m \in \mathbb{N}$ such that $a \in W_{\mu}^{-1,\infty}(0,1)$.

There is a small mistake in the proposition; we need the additional assumption that a vanishes on a neighbourhood of 0. This is actually noted in the proof and not at all that serious since we are mostly concerned about the behaviour of a near the point 1.

A related compactness result can also be verified:

Proposition 2.2.4. (Theorem 9 and Proposition 10) *Suppose that $a \in L^1_{loc}$ is radial, a vanishes on a neighbourhood of* 0 *and that a satisfies (2.5). Then $a \in W_{\nu}^{-1,\infty}(0,1)$ and has a representation*

$$a = \sum_{j\in\{0,1\}} (-1)^j D^j b_j,$$

where

$$\operatorname{ess} \lim_{s\to 1^-} \sup_{s<r<1} \mu(r)^{-j} |b_j(r)| = 0.$$

It follows that T_a is compact on A^p for $1 < p < \infty$.

Fredholm theory for matrix-valued symbols on A^1. We discuss the Fredholm criteria for matrix-valued symbols on $(A^1)^N$. This is an extension of the results of [25] to the matrix-valued case. Denote by $\mathcal{W}(z) := 1+\log(1/(1-|z|))$ the logarithmic weight on the unit disk. For $a \in L^1(\mathbb{D})$ and $r \in (0,1)$ we define the r-mean-oscillation $MO_r(a)$ by

$$MO_r(a)(z) = \frac{1}{|B(z,r)|} \int_{B(z,r)} |a(w) - a_{B(z,r)}| dA(w).$$

Here by $a_{B(z,r)}$ we mean the average of a over the Bergman disk $B(z,r)$. The value of r is not important, we can agree that $r = 1/2$ and just write $MO(f)$. We say that $a \in VMO_{\log}$ if

$$\lim_{|z|\to 1^-} \mathcal{W}(z) MO(f)(z) = 0.$$

We prove the following result, for more result about matrix-valued symbols, consult [E] and the corresponding section of this thesis.

Theorem 2.2.5. (Theorem 11) *Let a be a matrix-valued symbol $a = (a_{i,j})_{1\leq i,j\leq N}$ with $a_{i,j} \in C(\overline{\mathbb{D}}) \cap VMO_{\log}$. Then the Toeplitz operator T_a is Fredholm on $(A^1)^N$ if and only if $\det a(z) \neq 0$ on the boundary $\partial\mathbb{D}$. In this case*

$$\operatorname{Ind} T_a = -\operatorname{Ind} det(a)_1.$$

2.3. Article C: Toeplitz operators with distributional symbols on Fock spaces

In this paper we establish in the Fock space setting results analogous to those in [A]. Apart from dealing with symbols that are not necessarily functions this approach gives new results about unbounded function symbols. The idea behind this paper is quite similar to that of [A]. However, some additional difficulties emerge from the fact that the functions are defined on an unbounded set, the whole of $\mathbb{C}$.

The symbol classes $W_\omega^{-m,\infty}$. Our aim is to find a suitable class of distributions, much like in [A]. Let $\omega : \mathbb{C} \to \mathbb{R}$ be the weight-function defined as $\omega(z) = 1 + |z|$. A reasoning similar to that of [A] leads us to consider the following class of distributions.

Definition 2.3.1. Let $m \in \mathbb{N}$. The weighted distributional Sobolev space $W_\omega^{-m,\infty}$ consists of those distributions $a \in \mathcal{D}'$ that admit a representation

$$a = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.7}$$

where each b_α belongs to L_α^∞; that is, $\|b_\alpha; L_\alpha^\infty\| := \|\omega^{|\alpha|} b_\alpha\|_\infty < \infty$.

The derivative D^α is understood in the distributional sense. We can make $W_\omega^{-m,\infty}$ a Banach space by using the norm

$$\|a; W_\omega^{-m,\infty}\| = \inf \max_{|\alpha| \le m} \|b_\alpha; L_\alpha^\infty\|, \tag{2.8}$$

where the infimum is taken over all possible representations for a.

Integral estimates. The proof for the main result of this paper is analogous to that of [A]. However, there are some difficulties, which can be overcome with the help of the following lemma.

Lemma 2.3.2. (Lemma 3.2, Corollary 3.3 and Lemma 3.4) *Let $k \in \mathbb{N}$ and define the operators T_k, S_k and the mapping T_k' by*

$$\begin{aligned}
T_k f(z) &= z^k \int_{\mathbb{C}} f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2} b_\alpha(w) dA(w) \ , \ z \in \mathbb{C}; \\
T_k' f(z) &= \int_{\mathbb{C}} \omega(z)^k \omega(w)^{-k} |f(w) e^{\gamma z \bar{w}}| e^{-\gamma |w|^2} dA(w) \ , \ z \in \mathbb{C}; \\
S_k f(z) &= \omega(z)^{-k} f^{(k)}(z).
\end{aligned}$$

If $|\alpha| \ge k$ and $b_\alpha \in L_\alpha^\infty$, we have $\|T_k : L_\gamma^p \to F_\gamma^p\| \le C_1 \|b_\alpha; L_\alpha^\infty\|$, $\|T_k' f; L_\gamma^p\| \le C_2 \|f; L_\gamma^p\|$ and S_k is bounded $F_\gamma^p \to L_\gamma^p$.

Boundedness and compactness of Toeplitz operators with distributional symbols. The definition of Toeplitz operator in analogous to that of [A]. However, the reader should take notice how the Gaussian measure is taken into account in the following definition.

Definition 2.3.3. Suppose $a \in W_\omega^{-m,\infty}$ for some $m \in \mathbb{N}$. The Toeplitz operator T_a is then defined as

$$T_a f(z) := \sum_{|\alpha|\leq m} \int_{\mathbb{C}} D^\alpha (f(w)e^{\gamma z\bar{w}} e^{-\gamma|w|^2}) b_\alpha(w) dA(w) \ , \ z \in \mathbb{C}. \tag{2.9}$$

Here (b_α) is a representation for a.

The boundedness and compactness theorems come out as follows.

Theorem 2.3.4. (Theorem 4.1) *Suppose $a \in W_\omega^{-m,\infty}$ for some $m \in \mathbb{N}$. Then the Toeplitz operator T_a is bounded on F_γ^p for $\gamma > 0$ and $1 \leq p \leq \infty$. Moreover, there exists $C := C(m, p, \gamma) > 0$ such that*

$$\|T_a : F_\gamma^p \to F_\gamma^p\| \leq C\|a; W_\omega^{-m,\infty}\|.$$

Theorem 2.3.5. (Theorem 5.2) *Suppose $a \in W_\omega^{-m,\infty}$ for some $m \in \mathbb{N}$ and a has a representation*

$$a = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha b_\alpha,$$

where each b_α satisfies

$$\text{ess} \lim_{r\to\infty} \sup_{r<|z|} \omega^{|\alpha|}(z)|b_\alpha(z)| = 0.$$

Then T_a is compact on F_γ^p for $\gamma > 0$ and $1 \leq p \leq \infty$.

2.4. Article D: Toeplitz operators on Bloch-type spaces and classes of weighted Sobolev distributions

We study boundedness and compactness of Toeplitz operators acting between the generalized Bloch-spaces of the unit disk. The article deals with operators $\mathcal{B}_d \to \mathcal{B}_{d'}$ with $0 < d, d' < \infty$; to avoid repetition, this summary is only about Toeplitz operators on the classical Bloch space.

Symbol classes $\mathcal{LY}_0^m$ and $\mathcal{LV}_0^m$. When dealing with Bloch space operators, it is natural to consider logarithmically weighted symbol classes. We will use the weight functions

$$\nu(z) = 1 - |z|^2 \text{ and } \ell(z) = 1 + |\log(\nu(z))|.$$

Definition 2.4.1. Let $m \in \mathbb{N}$. The weighted distributional Sobolev space $\mathcal{LY}_0^m$ consists of those distributions $a \in \mathcal{D}'$ that admit a representation

$$a = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.10}$$

where each b_α belongs to $\mathcal{L}L^\infty_{-|\alpha|}$; that is, $\|b_\alpha; \mathcal{L}L^\infty_{-|\alpha|}\| := \|\ell\nu^{-|\alpha|} b_\alpha\|_\infty < \infty$.

We note that in the actual paper we need several other distributional classes, in which the logarithmic weight might, or might not be present. However, in each case the duality treatment and density results are similar to those of [A, B, C].

To shorten the notation, we also introduce the classes $\mathcal{LV}_0^m$:

Definition 2.4.2. Let $m \in \mathbb{N}$. A distribution $a \in \mathcal{LY}_0^m$ is said to belong to $\mathcal{LV}_0^m$ if a has a representation

$$a = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha b_\alpha,$$

where each where each b_α belongs to $\mathcal{L}L^\infty_{-|\alpha|}$ and

$$\text{ess}\lim_{r \to 1^-} \sup_{r<|z|<1} \ell(z)\nu(z)^{-|\alpha|}|b_\alpha(z)| = 0.$$

The class $\mathcal{LY}_0^m$ is just a logarithmic version of the class $W_\nu^{-m,\infty}$ of [A]. The class $\mathcal{LV}_0^m$ implies behaviour similar to what happens in Theorem 2.1.5, for instance.

Boundedness and compactness of Toeplitz operators on the Bloch space. The definition for Toeplitz operator with symbol in $\mathcal{LY}_0^m$ is the same as in Definition 2.1.3. We list here the main results for $T_a : \mathcal{B} \to \mathcal{B}$.

Theorem 2.4.3. (Part (2) of Corollary 3.8) *Suppose $a \in \mathcal{D}'$ belongs to $\mathcal{LY}_0^m$ for some $m \in \mathbb{N}$. Then T_a is bounded on the Bloch space and there exists a $C := C_m$ such that*

$$\|T_a : \mathcal{B} \to \mathcal{B}\| \le C\|a; \mathcal{LY}_0^m\|.$$

Theorem 2.4.4. (Part (2) of Corollary 4.7) *Suppose $a \in \mathcal{D}'$ belongs to $\mathcal{LV}_0^m$ for some $m \in \mathbb{N}$. Then T_a is compact on the Bloch space.*

It can be shown that the symbol class $\mathcal{LY}_0^m$ fails to contain constants functions, which is naturally a serious drawback, since a Toeplitz operator with contant symbol is just a multiple of the (obviously bounded) identity operator. This inconvenience can be partially fixed by studying logarithmic BMO_∂.

Let $f \in L^\infty$. Denote by $\hat{f}$ the averaging function:

$$\hat{f}(z) = |B(z,r)|^{-1} \int_{B(z,r)} f(w)dA(w),$$

where $B(z,r)$ is a Bergman disk.

In what follows, the choice of $r \in (0,1)$ is not important; just agree that $r = 1/2$, for instance.

Definition 2.4.5. Let $f \in L^\infty$. We say that f belongs to the logarithmic BMO_∂ ($f \in BMO_{\log}$) if

$$\|f; BMO_{\log}\| := \sup_{z \in \mathbb{D}} \ell(z)|D(z,r)|^{-1} \int_{D(z,r)} |f(w) - \hat{f}(z)|dA(w) < \infty.$$

We manage to improve our main Theorem 2.4.3 by the following.

Theorem 2.4.6. *Let $a \in \mathcal{D}'$ be a member of $\mathcal{Y}_0^m$ for some $m \in \mathbb{N}$. Assume moreover, that there exists $m' \in \mathbb{N}$ such that a has a representation*

$$a = \sum_{|\alpha| \le m'} D^\alpha b_\alpha,$$

where $b_\alpha/\nu^{|\alpha|} \in BMO_{\log}$ *for each* α*. Then* T_a *is bounded on* $\mathcal{B}$*.*

The Bergman space A^1. By duality, we also establish theory of distributional symbols for the Bergman space A^1.

Corollary 2.4.7. *Suppose* $a \in \mathcal{D}'$ *belongs to* $\mathcal{LY}_0^m$ *for some* $m \in \mathbb{N}$*. Then* T_a *is bounded on* A^1 *and there exists a constant* $C := C_m > 0$ *such that*

$$\|T_a : A^1 \to A^1\| \leq C\|a; \mathcal{LY}_0^m\|.$$

Corollary 2.4.8. *Suppose* $a \in \mathcal{D}'$ *belongs to* $\mathcal{LV}_0^m$ *for some* $m \in \mathbb{N}$*. Then* T_a *is compact on* A^1*.*

Corollary 2.4.9. *Let* $a \in \mathcal{D}'$ *be a member of* $\mathcal{Y}_0^m$ *for some* $m \in \mathbb{N}$*. Assume moreover, that there exists* $m' \in \mathbb{N}$ *such that* a *has a representation*

$$a = \sum_{|\alpha| \leq m'} D^\alpha b_\alpha,$$

where $b_\alpha/\nu^{|\alpha|} \in BMO_{\log}$ *for each* α*. Then* T_a *is bounded on* A^1*.*

2.5. Article E: A note on the Fredholm properties of Toeplitz operators on weighted Bergman spaces with matrix-valued symbols

We deal with Fredholm theory for Toeplitz operators having matrix-valued symbols. There are several results on the Hardy space setting, but the Bergman space theory seems still relatively incomplete. The reason might be, that except for some simple classes (such as $C(\overline{\mathbb{D}})$), the reduction of the matrix-valued case to the scalar-valued case is more complicated. Probably the best reference for the matrix-valued symbols on Hardy spaces is [8]. For the Bergman space we refer the reader to [11].

The weighted Bergman spaces of the unit ball. We work on the weighted Bergman spaces $A^p_\alpha(\mathbb{B}_n)$ on the unit ball of $\mathbb{C}^n$ with $\alpha > -1$ and $p \in (1, \infty)$. The Bergman projection $L^p_\alpha(\mathbb{B}_n) \to A^p_\alpha(\mathbb{B}_n)$ is then defined as

$$P_\alpha f(z) = \int_{\mathbb{B}_n} \frac{f(w)dA_\alpha(w)}{(1 - \langle z, w\rangle)^{n+1+\alpha}},$$

where $dA_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dA(z)$, dA is the standard $2n$-dimensional Lebesgue measure and c_α is a constant which makes dA_α a probability measure.

If $a \in L^\infty(\mathbb{B}_n)$, then the Toeplitz operator T_a is defined as

$$T_a f(z) = P_\alpha(af) \ , \ f \in A^p_\alpha.$$

Scalar-valued symbols. The symbols classes under consideration are $C := C(\overline{\mathbb{B}}_n)$, $H := C(\overline{\mathbb{B}}_n) + H^\infty(\mathbb{B}_n)$ and $Q := VMO_\partial(\mathbb{B}_n) \cap L^\infty(\mathbb{B}_n)$ and their matrix-valued counterparts: $C_N := C_{N\times N}$, $H_N := H_{N\times N}$ and $Q_N := Q_{N\times N}$, respectively. We will omit the dimension n from the notation; it is always the same as the n in $A^p_\alpha(\mathbb{B}_n)$. Both n and N are assumed to be natural numbers throughout this section. The Fredholm conditions for scalar-valued C and H are known, see for instance [10, 31] and the section of this thesis dealing with Fredholm theory. In our paper, we prove:

Theorem 2.5.1. (Theorem 3) *Let* $1 < p < \infty$, $n = 1$, $\alpha > -1$ *and* $a \in H$. *Then* T_a *is Fredholm on* $A^p_\alpha(\mathbb{D})$ *if and only if* a *is bounded away from zero near the boundary; that is, there exists* $\epsilon > 0$ *and* $r \in (0,1)$ *such that*

$$|a(z)| > \epsilon \ , \ \text{when } r < |z| < 1.$$

Moreover, the index of T_a *then satisfies*

$$\operatorname{Ind} T_a = -\operatorname{Ind} a_R,$$

for all $R \in (r, 1)$.

We recall that for $n > 1$, the index of T_a is known to be zero whenever $a \in H$ and T_a is Fredholm. We establish a similar result for a scalar-valued $a \in Q$. This is achieved by using the Berezin transform and the results in [32, 24].

Theorem 2.5.2. (Theorem 5) *Let* $1 < p < \infty$, $\alpha > -1$ *and* $a \in Q$. *Then* T_a *is Fredholm on* $A^p_\alpha(\mathbb{B}_n)$ *if and only if the Berezin transform of* a *is bounded away from zero near the boundary of* $\mathbb{B}_n$*; that is, there exists* $\epsilon > 0$ *and* $r \in (0,1)$ *such that*

$$|\tilde{a}(z)| > \epsilon \ , \ \text{when } r < |z| < 1.$$

If the above condition is satisfied and $n > 1$, *then the index of* T_a *is zero; if* $n = 1$, *then the index of* T_a *satisfies*

$$\operatorname{Ind} T_a = -\operatorname{Ind} \tilde{a}_R,$$

for all $R \in (r, 1)$.

Matrix-valued symbols. Theorems concerning matrix-valued symbols can often be reduced to theorems about scalar-valued symbols. In the setting of Hardy spaces, this is again settled, see [8]. The Bergman space theory is more difficult, mostly due to lack of boundary-values and the behaviour of the commutator $T_aT_b - T_bT_a$. The criteria for Fredholmness are still possible to verify:

Theorem 2.5.3. (Theorem 7) *Let* $1 < p < \infty$, $N \geq 2$ *and* $\alpha > -1$.

(i) If $a \in C_N$, *then* T_a *is Fredholm on* $(A^p_\alpha(\mathbb{B}_n))^N$ *if and only if* $\det a(z) \neq 0$ *on* $\partial\mathbb{B}_n$*; if in addition* $n = 1$, *we have*

$$\operatorname{Ind} T_a = \operatorname{Ind} T_{\det a} = -\operatorname{Ind}(\det a)_1.$$

(ii) If $a \in H_N$, *then* T_a *is Fredholm on* $(A^p_\alpha(\mathbb{B}_n))^N$ *if and only if* $\det a(z)$ *is bounded away from zero near the boundary.*

(iii) If $a \in Q_N$, *then* T_a *is Fredholm on* $(A^p_\alpha(\mathbb{B}_n))^N$ *if and only if* $\tilde{A}(z)$ *is bounded away from zero near the boundary, where* $A(z) = \det a(z)$.

We were unable to verify the index-formula for the classes H_N and Q_N if $N \neq 1$. We managed to prove a partial result:

Theorem 2.5.4. (Theorem 8) *Let* $1 < p < \infty$, $N \geq 2$ *and* $\alpha > -1$ *and* $n = 1$. *Suppose that either* $a \in H_N$ *or* $a \in Q_N$, T_a *is Fredholm on* $(A^p_\alpha(\mathbb{D}))^N$ *and that at least one of the following holds:*

(i) The scalar-valued Toeplitz operators $T_{a_{i,j}}$ *and* $T_{a_{k,l}}$ $1 \leq i, j, k, l \leq N$ *commute modulo trace class operators. Here* $a = (a_{i,j})$.

(ii) The operator T_{a_k} *is Fredholm on* $(A^p_\alpha(\mathbb{D}))^k$ *for each* $1 \leq k \leq N$, *where* $a_k = (a_{i,j})_{1 \leq i,j \leq k}$.

Then the index formula

$$\operatorname{Ind} T_a = \operatorname{Ind} T_{\det a}$$

holds.

3. Notes

3.1. Complementary material

The definition for weighted Sobolev spaces of distributions works well if one already has a suitable representation for a distribution $a \in \mathcal{D}'$. However, in general such representation might be problematic to find. The following proposition gives another method by the use of duality. We formulate the result only for the symbol class $\mathcal{Y}_0^m$; a similar result is true for all the distributional Sobolev classes presented in this thesis.

Proposition 3.1.1. (Proposition 2.5 of [D]) *Suppose $a \in \mathcal{D}'$. Then $a \in \mathcal{Y}_0^m$ if and only if there exists a positive constant C such that*

$$|\langle \varphi, a \rangle| \leq C \|\varphi; W_\nu^{m,1}\|$$

for every compactly supported test function φ.

For a compactly supported distribution a it is natural to define the Toeplitz-type operator S_a by the formula

$$S_a f(z) = \langle f(w)(1 - z\bar{w})^{-2}, a \rangle_w.$$

One might argue whether this definition is consistent with Definition 2.1.3 given in this thesis. We have shown the following (again, the same is true for every Sobolev class under consideration):

Proposition 3.1.2. (Proposition 4.2 of [D]) *Suppose a is a compactly supported distribution. Then there exists an $m \in \mathbb{N}$ such that $a \in \mathcal{Y}_0^m$. Moreover*

$$S_a = T_a,$$

in particular the operator S_a is a Toeplitz operator in the sense of Definition 2.1.3.

We would also like to note that, in the Fock space setting, it is possible to define a Toeplitz operator T_a, where $a \in \mathcal{S}'$ is a tempered distribution. The definition is given by the formula

$$T_a f(z) = \langle f(w) e^{\gamma z \bar{w}} e^{-\gamma |w|^2}, a \rangle_w,$$

where $\langle \cdot, \cdot \rangle$ stands for the dual bracket in $\langle \mathcal{S}, \mathcal{S}' \rangle$. The definition is then more general than ours, but boundedness and compactness questions become more complicated.

3.2. Errata

There is a mistake in proposition 7 of [B]; an additional assumption on vanishing of a on a neighbourhood of 0 is needed. This mistake is not that serious since this only means perturbating the operator by a compact operator, thus having no effect on boundedness.

The example 1 of [B] is also slightly misleading; it would be more correct to assume that h is a linear form on the space of real-analytic functions on $\mathbb{D}$, instead of being a (possibly discontinuous) linear form on C^∞. The example presented is, in fact, of the former type and correct.

A misprint appears a couple of times in article [D]. The proofs of Proposition 4.2, Lemma 4.3 begin with "Proofs of Theorems 3.5-3.7", which is incorrect, as the proofs are those of the aforementioned proposition and lemma. Similarly, the proofs of Lemma 5.3 and Theorem 5.4 begin with "Proof of Theorems 4.4-4.6".

Also, in page 114 of [D], the chain of inclusions is not correct. The rightmost space should be A^1_{d-1}; this claim does not hold, in general, for $p > 1$. This mistake has no effect on the outcome of the paper.

In [E] we prematurely conjecture that there should exist symbols a and b both polynomials of z and $\bar{z}$ such that the commutator

$$T_aT_b - T_bT_a$$

is not trace class on A^2. This conjecture was later proven false by Trieu Le, who contacted the author on this matter. The author wishes to thank Professor Le for pointing this out.

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